University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION 1

Directions: Do all 3 problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (45 points)
A surface charge of uniform density σ_0 Coul/m^2 is glued onto a spherical shell of radius R that is centered at the origin.

(**a**) (10 points)
Relative to ∞, find the potential V_0 at the origin.

Solution:
This part of the problem is spherically symmetric. Outside the shell, the potential is that of a point charge. Inside the shell, there is no charge, so the potential there is the same as at the shell's surface. Therefore

$$\begin{aligned} 4\pi\epsilon_0 V_0 &= \frac{q}{R} \\ &= \frac{4\pi R^2\sigma_0}{R} \\ V_0 &= \frac{\sigma_0 R}{\epsilon_0} \ . \end{aligned}$$

(**b**) (5 points)
How much work W was done to move the charge from ∞ to the shell?

Solution:

$$\begin{aligned} W &= \tfrac{1}{2}\int d\tau' \rho(\mathbf{r}')V(\mathbf{r}') \\ &= \tfrac{1}{2} qV(R) \\ &= \tfrac{1}{2} 4\pi R^2 \sigma_0 \frac{\sigma_0 R}{\epsilon_0} \\ W &= \frac{2\pi R^3 \sigma_0^2}{\epsilon_0} \ . \end{aligned}$$

(**c**) (10 points)
The shell is now split along its "equator" into two hemispheres, and the south hemisphere is thrown away. Find the new potential $V_{1/2}$ at the origin.

Solution:
We could have obtained the answer to (**a**) by doing the integral

$$4\pi\epsilon_0 V_0 = \int d\tau' \frac{\rho(\mathbf{r}')}{r'} \ .$$

Now, with half of the shell removed, the integral is half as big. Therefore

$$V_{1/2} = \frac{V_0}{2} = \frac{\sigma_0 R}{2\epsilon_0} \ .$$

(**d**) (20 points)
For the conditions of part (**c**), calculate the potential V_N at the "north pole" $(0,0,R)$.

Solution:
Now we need actually to do an integral. Consider a ring $d\theta'$ of charge, where θ' is the angle measured from the north pole. This ring has area $da' = 2\pi R^2 \sin\theta' d\theta'$ and is located a distance $r' = 2R\sin\frac{\theta'}{2}$ from the north pole. The contribution from this ring to the potential at the north pole is

$$\begin{aligned} 4\pi\epsilon_0 \, dV_N &= \frac{\sigma_0 da'}{r'} \\ &= \frac{\sigma_0 2\pi R^2 \sin\theta'}{2R\sin\frac{\theta}{2}} d\theta' \\ &= \frac{\pi R\sigma_0 \sin\theta'}{\sin\frac{\theta'}{2}} d\theta' \ . \end{aligned}$$

Substituting

$$\begin{aligned}\sin\theta' &= \sin\left(\tfrac{\theta'}{2}+\tfrac{\theta'}{2}\right)\\ &= 2\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\ ,\end{aligned}$$

we have

$$\begin{aligned}4\pi\epsilon_0\, dV_N &= \frac{\pi R\sigma_0 2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}}{\sin\frac{\theta'}{2}}d\theta'\\ &= 2\pi R\sigma_0\cos\tfrac{\theta'}{2}2d\tfrac{\theta'}{2}\ .\end{aligned}$$

Integrating over $0<\theta'<\frac{\pi}{2}$,

$$\begin{aligned}4\pi\epsilon_0 V_N &= 4\pi R\sigma_0\int_0^{\pi/4}\cos\tfrac{\theta}{2}d\tfrac{\theta}{2}\\ &= 4\pi R\sigma_0\sin\tfrac{\pi}{4}\\ V_N &= \frac{\sigma_0 R}{\sqrt{2}\epsilon_0}\ .\end{aligned}$$

As a check, if we had integrated θ' all the way to π, including both hemispheres, we would have recovered the answer to (**a**).

Problem 2. (25 points)
A point charge q is held at a distance z above an infinite conducting plane that is grounded ($V=0$). Calculate the surface charge density σ_s on the plane at a distance $s\gg z$ from the charge. Accuracy to lowest nonvanishing order in z/s is sufficient.

Solution:
For $z>0$, the effect of the charge that is induced on the conducting plane is the same as that of an image charge $-q$ a distance z below the plane. Together the physical charge and the image charge form a physical dipole with moment $\mathbf{p}=\hat{z}q2z$. At a cylindrical radius $s\gg z$, the field of the physical dipole is approximately the same as that of an ideal dipole:

$$\begin{aligned}\frac{4\pi\epsilon_0 r^3}{p}\mathbf{E} &= 3\hat{r}(\hat{r}\cdot\hat{p})-\hat{p}\\ &= 3\hat{s}(\hat{s}\cdot\hat{z}=0)-\hat{z}\\ 4\pi\epsilon_0\mathbf{E} &= -\hat{z}\frac{2qz}{s^3}\ .\end{aligned}$$

The surface charge density on the conductor is just $\epsilon_0 E_z$, so

$$\sigma_s = -\frac{qz}{2\pi s^3}\ .$$

Apart from factors of order unity, the answer $-qz/s^3$ could be guessed. Since a dipole is involved, the result must be proportional to its moment and thus to z. Given that, $-qz/s^3$ is the only acceptable combination of the available variables that has the dimensions of a surface charge density. This argument is worth some part credit.

This problem could also be approached by considering separately the electric fields from the physical and image charges, expanding them in powers of z/s, and retaining the leading terms that do not cancel. If you attempted to do this and fouled it up, you shouldn't expect excessive part credit, as such an approach doesn't require excessive physical insight.

Problem 3. (30 points)
A thin phonograph record is composed of a material that has a uniform volume charge density; the total charge is Q. The record has radius R and rotates on a turntable at angular velocity $\vec{\omega}$. Calculate the magnetic field at the center of the record.

Solution:
Again we need to do an integral. Define $\hat{z} \equiv \hat{\omega}$ and s to be the (cylindrical) radius. Consider an element ds of the record, located a distance s from its center. The charge dQ on this element is

$$\begin{aligned} dQ &= Q\frac{2\pi s\, ds}{\pi R^2} \\ &= \frac{2Qs}{R^2} ds\ . \end{aligned}$$

This charge rotates once every $2\pi/\omega$ seconds, so the element carries a current

$$\begin{aligned} dI &= \frac{\omega}{2\pi}\frac{2Qs}{R^2} ds \\ &= \frac{\omega Q s}{\pi R^2} ds\ . \end{aligned}$$

When one applies the Biot-Savart law, one finds that a circular loop of current I_0 and radius s_0 has a central field equal to $\mu_0 I_0/2s_0$. Therefore the contribution of the record element ds to the central magnetic field is

$$\begin{aligned} d\mathbf{B} &= \hat{z}\frac{\mu_0}{2s} dI \\ &= \hat{z}\frac{\mu_0}{2s}\frac{\omega Q s}{\pi R^2} ds \\ &= \hat{z}\frac{\mu_0 \omega Q}{2\pi R^2} ds \\ \mathbf{B} &= \hat{z}\frac{\mu_0 \omega Q}{2\pi R^2}\int_0^R ds \\ \mathbf{B} &= \hat{\omega}\frac{\mu_0 \omega Q}{2\pi R}\ . \end{aligned}$$